L2 - Path Integral of Lattice Discretized Scalar Field Theory

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Path integral

$$\hat{H} = \hat{K} + \hat{V} \qquad \hat{K} = \sum_{\underline{x}} \frac{a^3}{2} \hat{\pi}^2(\underline{x}) \qquad \hat{V} = V(\hat{\varphi}) = \sum_{\underline{x}} a^3 \left\{ \frac{1}{2} (\partial_k^f \hat{\varphi})^2 + \frac{m^2}{2} \hat{\varphi}^2 + \frac{\lambda}{4!} \hat{\varphi}^4 \right\}(\underline{x})$$

Trotter formula
$$e^{-T\hat{H}} = \left(e^{-\frac{T}{M}\hat{H}} \right) = \lim_{M\to\infty} \left(e^{-\frac{T}{M}\hat{K}} e^{-\frac{T}{M}\hat{V}} \right)^M$$
[strong limit]

T = total time

$\tau = T/M$ = time step $\xrightarrow{M\to\infty} 0$

$$e^{-\tau\hat{H}} = e^{-\tau(\hat{K}+\hat{V})} \overset{\tau\to 0}{=} 1 - \tau(\hat{K}+\hat{V}) + O(\tau^2) = \{1-\tau\hat{K}\}\{1-\tau\hat{V}\} + O(\tau^2) = e^{-\tau\hat{K}} e^{-\tau\hat{V}} e^{O(\tau^2)}$$

Transfer matrix $\hat{T}(\tau) \overset{\text{def}}{=} e^{-\tau K} e^{-\tau V}$ (approximated evolution operator for a single time step)

Trotter formula (2) $$e^{-T\hat{H}} = \lim_{\substack{M\to\infty \\ \tau = T/M}} \left[\hat{T}(\tau) \right]^M$$

Path integral (v2)

$$\hat{H} = \hat{K} + \hat{V} \qquad \hat{K} = \sum_{\underline{x}} \frac{a^3}{2} \hat{\pi}^2(\underline{x}) \qquad \hat{V} = V(\hat{\varphi}) = \sum_{\underline{x}} a^3 \left\{ \frac{1}{2} (\partial^f_k \hat{\varphi})^2 + \frac{m^2}{2} \hat{\varphi}^2 + \frac{\lambda}{4!} \hat{\varphi}^4 \right\}(\underline{x})$$

Trotter formula

$$e^{-T\hat{H}} = \left(e^{-\frac{T}{M}\hat{H}}\right) = \lim_{M\to\infty} \left(e^{-\frac{T}{M}\hat{K}} e^{-\frac{T}{M}\hat{V}}\right)^M \qquad \text{[strong limit]}$$

$$= \lim_{M\to\infty} e^{\frac{T}{2M}\hat{V}} \left(e^{-\frac{T}{2M}\hat{V}} e^{-\frac{T}{M}\hat{K}} e^{-\frac{T}{2M}\hat{V}}\right)^M e^{-\frac{T}{2M}\hat{V}} = \lim_{M\to\infty} \left(e^{-\frac{T}{2M}\hat{V}} e^{-\frac{T}{M}\hat{K}} e^{-\frac{T}{2M}\hat{V}}\right)^M$$

T = total time

$\tau = T/M$ = time step $\xrightarrow{M\to\infty} 0$

$$e^{-\tau\hat{H}} = e^{-\tau(\hat{K}+\hat{V})} \overset{\tau\to 0}{=} 1 - \tau(\hat{K}+\hat{V}) + O(\tau^2) = \{1-\tau\hat{K}\}\{1-\tau\hat{V}\} + O(\tau^2) = e^{-\tau\hat{K}} e^{-\tau\hat{V}} e^{O(\tau^2)}$$

Transfer matrix $\hat{T}(\tau) \overset{\text{def}}{=} e^{-\frac{\tau}{2}\hat{V}} e^{-\tau\hat{K}} e^{-\frac{\tau}{2}\hat{V}}$ $\rightsquigarrow$ approximated evolution operator for a single time step

Trotter formula (2) $\quad e^{-T\hat{H}} = \lim_{\substack{M\to\infty \\ \tau=T/M}} \left[\hat{T}(\tau)\right]^M$

properties $\hat{T}^+ = \hat{T}$, $\hat{T} \geq 0$ $\quad\Vert\quad$ not satisfied if you choose $\hat{T} = e^{-\tau\hat{V}} e^{-\tau\hat{K}}$

(in particular $\hat{T}$ is diagonalizable)

Path-integral formula for the transition amplitude between eigenvectors of $\hat{\varphi}(\underline{x})$

$$\langle \varphi_T | e^{-T\hat{H}} | \varphi_0 \rangle = \lim_{\substack{M\to\infty \\ \tau = T/M}} \langle \varphi_T | [\hat{T}(\tau)]^M | \varphi \rangle$$

$$\langle \varphi_T | [\hat{T}(\tau)]^M | \varphi_0 \rangle = \langle \varphi_T | \overbrace{\hat{T}(\tau) \cdots \hat{T}(\tau)\, \hat{T}(\tau)}^{M \text{ times}} | \varphi_0 \rangle$$

completeness relation

$$d\varphi = \prod_{\underline{x}} d\varphi(\underline{x})$$

$$I = \int d\varphi_\tau \, |\varphi_\tau \rangle\langle \varphi_\tau | \quad \text{(at time } t = \tau)$$

$$I = \int d\varphi_{2\tau} \, |\varphi_{2\tau} \rangle\langle \varphi_{2\tau} | \quad \text{(at time } t = 2\tau)$$

$$= \int d\varphi_\tau \, d\varphi_{2\tau} \cdots d\varphi_{T-\tau} \, \langle \varphi_T | \hat{T}(\tau) | \varphi_{T-1} \rangle \cdots \langle \varphi_{2\tau} | \hat{T}(\tau) | \varphi_\tau \rangle \langle \varphi_\tau | \hat{T}(\tau) | \varphi_0 \rangle$$

$$= \int \underbrace{\left[\prod_{t=\tau,\dots,T-\tau} d\varphi_\tau \right]}_{\text{a temporal lattice emerges from this formulae}} \left[\prod_{t=0,\tau,\dots,T-\tau} \underbrace{\langle \varphi_{t+\tau} | \hat{T}(\tau) | \varphi_t \rangle}_{\text{this can be analytically calculated!}} \right]$$

let's do it...

$$\langle \varphi_{t+\tau} | \hat{T}(\tau) | \varphi_t \rangle \overset{①}{=} \langle \varphi_{t+\tau} | e^{-\frac{\tau}{2}\hat{V}} e^{-\tau\hat{K}} e^{-\frac{\tau}{2}\hat{V}} | \varphi_t \rangle$$

$$\overset{②}{=} e^{-\frac{\tau}{2}V(\varphi_{t+\tau})} e^{-\frac{\tau}{2}V(\varphi_t)} \underbrace{\langle \varphi_{t+\tau} | e^{-\tau\hat{K}} | \varphi_t \rangle}_{\text{Euclidean Kernel (problem 00/02)}}$$

$$\overset{③}{=} e^{-\tau\frac{V(\varphi_{t+\tau})+V(\varphi_t)}{2}} \int \left[\prod_{\underline{x}} \frac{d\pi(\underline{x})}{2\pi}\right] \langle \varphi_{t+\tau} | e^{-\tau\hat{K}} | \pi \rangle\langle \pi | \varphi_t \rangle$$

$$\overset{④}{=} e^{-\tau\frac{V(\varphi_{t+\tau})+V(\varphi_t)}{2}} \int \left[\prod_{\underline{x}} \frac{d\pi(\underline{x})}{2\pi}\right] e^{-\tau\sum_{\underline{x}}\frac{a^3}{2}\pi^2(\underline{x})} e^{i\sum_{\underline{x}} a^3 \pi(\underline{x})[\varphi_t(\underline{x}) - \varphi_{t+\tau}(\underline{x})]}$$

$$\overset{⑤}{=} e^{-\tau\frac{V(\varphi_{t+\tau})+V(\varphi_t)}{2}} \prod_{\underline{x}} \left\{ \int \frac{d\pi}{2\pi} e^{-\frac{\tau a^3}{2}\pi^2 - i\tau a^3 \pi\, \partial_0^f \varphi_t(\underline{x})} \right\}$$

$$\overset{⑥}{=} e^{-\tau\frac{V(\varphi_{t+\tau})+V(\varphi_t)}{2}} \prod_{\underline{x}} \left\{ (2\pi\tau a^3)^{-1/2} e^{-\frac{\tau a^3}{2}[\partial_0^f \varphi_t(\underline{x})]^2} \right\}$$

$$= (2\pi\tau a^3)^{-N^3/2} \exp\left\{ -\tau \left[\underbrace{\sum_{\underline{x}} \frac{a^3}{2}(\partial_0^f \varphi_t)^2 + \frac{V(\varphi_{t+\tau})+V(\varphi_t)}{2}}_{\text{Euclidean discretized Lagrangian} \equiv L_E(t)} \right] \right\}$$

① Definition of $\hat{T}$

② $\hat{V} = V(\hat{\varphi})$

$e^{-\frac{\tau}{2}\hat{V}}|\varphi\rangle = e^{-\frac{\tau}{2}V(\varphi)}|\varphi\rangle$

③ $I = \int \left[\prod_{\underline{x}} \frac{d\pi(\underline{x})}{2\pi}\right] |\pi\rangle\langle\pi|$

④ $\left[\begin{array}{l} \hat{K}|\pi\rangle = \sum_{\underline{x}} \frac{a^3}{2}\pi^2(\underline{x})|\pi\rangle \\ \langle \pi | \varphi \rangle = \exp\left\{ i\sum_{\underline{x}} a^3 \varphi(\underline{x})\pi(\underline{x}) \right\} \end{array}\right.$

⑤ $e^{\sum_{\underline{x}} ⊛} = \prod_{\underline{x}} e^{⊛}$

$\partial_0^f \varphi_t(\underline{x}) \overset{\text{def}}{=} \frac{\varphi_{t+\tau}(\underline{x}) - \varphi_t(\underline{x})}{\tau}$

⑥ $\int_{-\infty} dx\, e^{-\frac{\alpha}{2}x^2 + ipx} = \sqrt{\frac{2\pi}{\alpha}}\, e^{-\frac{p^2}{2\alpha}}$

$\alpha \leftarrow \tau a^3 > 0$; $p \leftarrow -\tau a^3 \partial_0^f \varphi_t(\underline{x})$

$x \leftarrow \pi$

(problem 00/01)

Let's bring together the results of the previous two slides

$$\langle \varphi_T | [\hat{T}(\tau)]^M | \varphi_0 \rangle = \int \underbrace{\left[\prod_{t=\tau,\dots,T-\tau} d\varphi_t \right]}_{\text{notation} \equiv [d\varphi]_{(0,T)}} \prod_{t=0,\dots,T-\tau} \underbrace{\langle \varphi_{t+\tau} | \hat{T}(\tau) | \varphi_t \rangle}_{\text{calculation } (2\pi\tau a^3)^{-N^3/2} \exp\{-\tau L_E(t)\}}$$

$$= (2\pi\tau a^3)^{-\frac{N^3 M}{2}} \int [d\varphi]_{(0,T)} \exp\Big\{ - \underbrace{\sum_{t=0}^{T-\tau} \tau L_E(t)}_{\text{discretized Euclidean action} \equiv S^E_{[0,T)}(\varphi)} \Big\}$$

Path integral formula

Summary

$$\hat{H} = \sum_{\underline{x}} a^3 \left\{ \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\left(\partial_n^f \hat{\varphi}\right)^2 + \frac{m^2}{2}\hat{\varphi}^2 + \frac{\lambda}{4!}\hat{\varphi}^4 \right\} \equiv \hat{K} + \hat{V} \rightarrow$$ Hamiltonian (on 3d lattice)

$$\langle \varphi_T | e^{-T\hat{H}} | \varphi_0 \rangle = \lim_{\substack{M\to\infty \\ \tau = T/M}} \langle \varphi_T | [\hat{T}(\tau)]^M | \varphi_0 \rangle$$ with $$\hat{T}(\tau) = e^{-\frac{\tau}{2}\hat{V}} e^{-\tau\hat{K}} e^{-\frac{\tau}{2}\hat{V}}$$ ↳ transfer matrix

$$\langle \varphi_T | [\hat{T}(\tau)]^M | \varphi_0 \rangle = \left(2\pi\tau a^3\right)^{-\frac{N^3 M}{2}} \int [d\varphi]_{(0,T)}\, e^{-S_{[0,T)}(\varphi)} \rightarrow$$ (discretized) path-integral formula for transition amplitudes

with $$[d\varphi]_{(0,T)} = \prod_{t=\tau}^{T-\tau} \prod_{\underline{x}} d\varphi(t,\underline{x}) \rightarrow$$ path-integral measure (over all fields on 4d lattice)

$$S_{[0,T)}(\varphi) = \sum_{t=0}^{T-\tau} \tau \left\{ \sum_{\underline{x}} \frac{a^3}{2} \left(\partial_0^f \varphi\right)^2 + \frac{V(\varphi_{t+\tau}) + V(\varphi_t)}{2} \right\}$$ Euclidean action (on 4d lattice)

$\hat{H}$ is discretized on a 3d lattice with lattice spacing a and periodic boundary conditions

S is discretized on a 4d lattice:
* 3d spatial lattice with spacing a and periodic boundary conditions
* 1d temporal lattice with spacing τ and Dirichlet boundary conditions (φ_0 and φ_T are fixed)

Special choice: $a = \tau$ (symmetric lattice)

Path-integral formula for thermal partition function

$$Z = \mathrm{tr}\, e^{-TH} = \lim_{\substack{M\to\infty \\ \tau = T/M}} \mathrm{tr}\left\{ \hat{T}^M(\tau) \right\}$$

notice T = Euclidean time = (temperature)$^{-1}$

$$\mathrm{tr}\left\{ \hat{T}^M(\tau) \right\} = \int d\varphi_0 \, \langle \varphi_0 | \hat{T}^M(\tau) | \varphi_0 \rangle = \int d\varphi_0 \, \langle \varphi_T | \hat{T}^M(\tau) | \varphi_0 \rangle \Big|_{\varphi_T = \varphi_0}$$

$$= (2\pi\tau a^3)^{-\frac{N^3 T}{2}} \int d\varphi_0 \int [d\varphi]_{(0,T)} \, e^{-S_{[0,T)}(\varphi)} \Big|_{\varphi_T = \varphi_0}$$

$$= (2\pi\tau a^3)^{-\frac{N^3 M}{2}} \int [d\varphi]_{[0,T)} \, e^{-S_{[0,T)}(\varphi)} \Big|_{\varphi_T = \varphi_0}$$

Matrix elements $\langle \varphi_T | \hat{T}^M | \varphi_0 \rangle$ $\leadsto$ Dirichlet b.c. in time

Partition function $\mathrm{tr}\, \hat{T}^M$ $\leadsto$ Periodic b.c. in time

$$S_{[0,T)}(\varphi) = \sum_{t=0}^{T-\tau} \sum_{x} \frac{\tau a^3}{2} (\partial_0^f \varphi_t)^2 + \underbrace{\frac{1}{2} \sum_{t=0}^{T-\tau} \tau V(\varphi_t) + \frac{1}{2} \sum_{t=0}^{T-\tau} \tau V(\varphi_{t+\tau})}_{\text{equal terms because } \varphi_T = \varphi_0} = \sum_{t=0}^{T-\tau} \tau \left\{ \sum_{x} \frac{a^3}{2} (\partial_0^f \varphi_t)^2 + V(\varphi_t) \right\}$$

$$= \sum_{t=0}^{T-\tau} \sum_{x} \tau a^3 \left\{ \frac{1}{2} (\partial_\mu^f \varphi)^2 + \frac{m^2}{2} \varphi^2 + \frac{\lambda}{4!} \varphi^4 \right\}$$

Core concepts of thermal QFT

- A quantum mechanical system in thermal equilibrium with inverse temperature β is described by the statistical state / density matrix

$$\hat{\rho} = \frac{1}{Z} e^{-\beta \hat{H}} \qquad \text{with} \qquad \begin{array}{l} \hat{H} = \text{Hamiltonian} \\ Z = \operatorname{tr} e^{-\beta \hat{H}} = \text{partition function} \end{array} \qquad \left[\begin{array}{c}\text{canonical} \\ \text{ensemble}\end{array}\right]$$

- The thermal expectation value of a generic observable $\hat{A}$ is given by

$$\operatorname{tr}\{\hat{\rho}\hat{A}\} = \frac{1}{Z} \operatorname{tr}\{e^{-\beta H}\hat{A}\}$$

- inverse temperature $= \beta = T =$ Euclidean time

- The thermal n-point functions of fields is defined as

$$\operatorname{tr}\{\rho\, \hat{\varphi}(t_n, \underline{x}_n) \cdots \hat{\varphi}(t_2, \underline{x}_2)\, \hat{\varphi}(t_1, \underline{x}_1)\}$$

$$= \frac{1}{Z} \operatorname{tr}\{e^{-(T-t_n)} \hat{\varphi}(\underline{x}_n) e^{-(t_n - t_{n-1})H} \hat{\varphi}(\underline{x}_{n-1}) \cdots \hat{\varphi}(\underline{x}_2) e^{-(t_2 - t_1)H} \hat{\varphi}(\underline{x}_1) e^{-t_1 H}\}$$

$$\hat{\varphi}(t, \underline{x}) = e^{tH} \hat{\varphi}(\underline{x}) e^{-tH}$$

(Euclidean - Heisenberg picture)

In particular, we will often consider time-ordered n-pt functions, in this case $T > t_n > t_{n-1} > \ldots > t_2 > t_1 > 0$

Path-integral formula for thermal time-ordered n-point functions

$$\mathrm{tr}\{\rho\, \hat\varphi(t_n,\underline{x}_n)\cdots\hat\varphi(t_2,\underline{x}_2)\hat\varphi(t_1,\underline{x}_1)\}$$

$$\overset{(M\to\infty)}{=} \frac{1}{Z}\,\mathrm{tr}\{e^{-(T-t_n)}\hat\varphi(\underline{x}_n)e^{-(t_n-t_{n-1})H}\hat\varphi(\underline{x}_{n-1})\cdots\hat\varphi(\underline{x}_2)e^{-(t_2-t_1)H}\hat\varphi(\underline{x}_1)e^{-t_1H}\}$$

$$\hat T(\tau)^{-\frac{T-\tilde t_n}{\tau}} \quad (T-\tilde t_n>0) \qquad \hat T(\tau)^{\frac{\tilde t_n-\tilde t_{n-1}}{\tau}} \quad (\tilde t_n-\tilde t_{n-1}>0) \qquad \hat T(\tau)^{\frac{\tilde t_2-\tilde t_1}{\tau}} \quad (\tilde t_2-\tilde t_1>0) \qquad \hat T(\tau)^{\frac{\tilde t_1}{\tau}} \quad (\tilde t_1>0)$$

$$\hat\varphi(\underline{x}_k) = \hat\varphi(\underline{x}_k)\int d\varphi_{\tilde t_k}\,|\varphi_{\tilde t_k}\rangle\langle\varphi_{\tilde t_k}|$$

$$= \int d\varphi_{\tilde t_k}\,|\varphi_{\tilde t_k}\rangle\,\varphi_{\tilde t_k}(\underline{x}_k)\,\langle\varphi_{\tilde t_k}|$$

$$\mathrm{tr}\{\circledast\} = \int d\varphi_0\,\langle\varphi_T|\circledast|\varphi_0\rangle\Big|_{\varphi_T=\varphi_0}$$

$$= \frac{(2\pi\tau a^3)^{-\frac{N^3M}{2}}}{Z}\int[d\varphi]_{[0,T)}\,e^{-S_{[0,T)}(\varphi)}\prod_{k=1}^{n}\varphi(\tilde t_k,\underline{x}_k)\Big|_{\varphi_T=\varphi_0}$$

↳ when using the formula for Z, the factor $(2\pi\tau a^3)^{-\frac{N^3M}{2}}$ simplifies

$M=T/\tau$ integer

t_k/τ is generally not an integer

$\tilde t_k$ is some point of the temporal lattice that approximates t_k (in the future I will omit the tilde)

Path-integral formula for thermal time-ordered n-point functions

(thermal QM expectation value)

$$\mathrm{tr}\left\{\rho\,\hat\varphi(t_n,\underline{x}_n)\cdots\hat\varphi(t_2,\underline{x}_2)\,\hat\varphi(t_1,\underline{x}_1)\right\} = \lim_{\substack{M\to\infty\\ \tau=T/M}} \underbrace{\frac{\mathrm{tr}\left\{\hat{T}(\tau)^{\frac{T-\tilde t_n}{\tau}}\,\hat\varphi(\underline{x}_n)\,\hat{T}(\tau)^{\frac{\tilde t_n-\tilde t_{n-1}}{\tau}}\cdots\hat\varphi(\underline{x}_2)\,\hat{T}(\tau)^{\frac{\tilde t_2-\tilde t_1}{\tau}}\,\hat\varphi(\underline{x}_1)\,\hat{T}(\tau)^{\tilde t_1/\tau}\right\}}{\mathrm{tr}\left\{\hat{T}(\tau)^{T/\tau}\right\}}}_{\circledast}$$

$$\circledast = \frac{\int[d\varphi]_{[0,T)}\,e^{-S_{[0,T)}(\varphi)}\prod_{k=1}^{n}\varphi(\tilde t_k,\underline{x}_k)\Big|_{\varphi_T=\varphi_0}}{\int[d\varphi]_{[0,T)}\,e^{-S_{[0,T)}(\varphi)}\Big|_{\varphi_T=\varphi_0}} \equiv \left\langle\prod_{k=1}^{n}\varphi(\tilde t_k,\underline{x}_k)\right\rangle$$

(path-integral expectation value)

$M\times N^3$ lattice with spacings τ, a periodic b.c. in space and time

Probabilistic interpretation: $\langle A(\varphi)\rangle = \int[d\varphi]\,A(\varphi)\,P(\varphi)$ with $P(\varphi)=\dfrac{e^{-S(\varphi)}}{\int[d\varphi']\,e^{-S(\varphi')}}$

If $S(\varphi)$ is real: $P(\varphi)\geq 0$ and $\int[d\varphi]\,P(\varphi)=1$ i.e. $P(\varphi)$ is a probability distribution on the space of fields defined on a 4d lattice

path integral expectation value $\equiv$ exp. value w.r.t. $P(\varphi)$

Summary (2)

$$\hat{H}_{QFT} = \int d^3x \left\{ \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}(\partial_n \hat{\varphi})^2 + \frac{m^2}{2}\hat{\varphi}^2 + \frac{\lambda}{4!}\hat{\varphi}^4 \right\}$$ QFT Hamiltonian

$$\hat{H} = \sum_{\underline{x}} a^3 \left\{ \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\left(\partial_n^f \hat{\varphi}\right)^2 + \frac{m^2}{2}\hat{\varphi}^2 + \frac{\lambda}{4!}\hat{\varphi}^4 \right\} \equiv \hat{K} + \hat{V}$$ Hamiltonian on 3d lattice

$$\hat{T}(\tau) = e^{-\frac{\tau}{2}\hat{V}}\, e^{-\tau \hat{K}}\, e^{-\frac{\tau}{2}\hat{V}}$$ Transfer matrix

$$S_I = \sum_{t \in I} \tau \left\{ \sum_{\underline{x}} \frac{a^3}{2} (\partial_0^f \varphi)^2 (t,x) + \frac{V(\varphi_t) + V(\varphi_{t+\tau})}{2} \right\}$$ Euclidean action (on 4d lattice)

$$[d\varphi]_I = \prod_{t \in I} \prod_{\underline{x}} d\varphi(t,\underline{x})$$ path-integral measure (over all fields on 4d lattice)

	matrix elements of Euclidean evolution operator	thermal expectation values of time-ordered product of fields
continuous time	$\langle \varphi_T \vert e^{-T\hat{H}} \vert \varphi_0 \rangle$	$\frac{1}{Z} \mathrm{tr} \left\{ e^{-T\hat{H}}\, \hat{\varphi}(t_n,\underline{x}_n) \cdots \hat{\varphi}(t_1,\underline{x}_1) \right\}$ $\quad T > t_n > \ldots > t_1 > 0$
discrete time	$\langle \varphi_T \vert \hat{T}^M \vert \varphi_0 \rangle$	$\frac{1}{Z_{\text{discr time}}} \mathrm{tr} \left\{ \hat{T}^{\frac{T-\tilde{t}_n}{\tau}} \hat{\varphi}(\underline{x}_n) \cdots \hat{\varphi}(\underline{x}_2) \hat{T}^{\frac{\tilde{t}_2-\tilde{t}_1}{\tau}} \hat{\varphi}(\underline{x}_1) \hat{T}^{\frac{\tilde{t}_1}{\tau}} \right\}$
path-integral representation	$(2\pi\tau a^3)^{-\frac{N^3 M}{2}} \int [d\varphi]_{(0,T)}\, e^{-S_{[0,T)}(\varphi)}$	$\frac{1}{N} \int [d\varphi]_{[0,T)}\, e^{-S_{[0,T)}(\varphi)} \prod_{k=1}^{n} \varphi(\tilde{t}_k, \underline{x}_k) \Big\vert_{\varphi_T = \varphi_0}$

The QFT limit(s)

In discretizing the theory, we have introduced 4 artificial parameters

$a =$ spatial lattice spacing
$\tau =$ temporal lattice spacing
$N =$ # of points in a spatial direction
$M =$ # of points in the temporal direction

$L = aN =$ length of sp. direction
$T = \tau M =$ length of temp. direction

QFT is obtained when $a, \tau \to 0$ and $N, M \to \infty$. These limits can be taken in different ways, e.g. [the $a \to 0$ limit requires renormalization! Later...]

$\lim_{a,\tau\to 0} \lim_{\substack{N,M\to\infty \\ a,\tau \text{ fixed}}}$ [lattice obs.] QFT $(T, L = \infty)$

$\lim_{\substack{a,\tau\to 0,\ N,M\to\infty \\ L,T \text{ fixed}}}$ [lattice obs.] finite-T finite-L QFT

$\lim_{\substack{a,\tau\to 0 \\ T \text{ fixed}}} \lim_{N\to\infty}$ [lattice obs.] finite-T QFT $(L = \infty)$

$\lim_{\substack{a,\tau\to 0 \\ L \text{ fixed}}} \lim_{M\to\infty}$ [lattice obs.] finite-L QFT $(T = \infty)$

Observables

Brainstorming (think of φ^4, QED, QCD, SM...)

Cross sections involving stable asymptotic particles or long-lived particles (e.g. μ)

experimentally measurable

theoretically well defined

What is a particle in QFT? What is a particle's mass? (NB: We call "particles" many things...)

We want to focus on stable asymptotic particles (not necessarily elementary)
electron, photon, proton, stable nuclei, stable atoms

theoretically ambiguous
[renormalization
IR-div.
gauge-dependent]